

# Revenue Gaps for Discriminatory and Anonymous Sequential Posted Pricing\*

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## Abstract

We consider the problem of selling a single item to one of  $n$  bidders who arrive sequentially with values drawn independently from identical distributions, and ask how much more revenue can be obtained by posting discriminatory prices to individual bidders rather than the same anonymous price to all of them. The ratio between the maximum revenue from discriminatory pricing and that from anonymous pricing is at most  $2 - 1/n$  for arbitrary distributions and at most  $1/(1 - (1 - 1/n)^n) \leq e/(e - 1) \approx 1.582$  for regular distributions, and these bounds can in fact be obtained by using one of the discriminatory prices as an anonymous one. The bounds are shown via a relaxation of the discriminatory pricing problem rather than virtual values and thus apply to distributions without a density, and they are tight for all values of  $n$ . For a class of distributions that includes the uniform and the exponential distribution we show the maximization of revenue to be equivalent to the maximization of welfare with an additional bidder, in the sense that both use the same discriminatory prices. The problem of welfare maximization is the well-known Cayley-Moser problem, and this connection can be used to establish that the revenue gap between discriminatory and anonymous pricing is approximately 1.037 for the uniform distribution and approximately 1.073 for the exponential distribution.

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# 1 Introduction

Assume that there is a single item for sale and a sequence of  $n$  bidders interested in the item arrive one by one with valuations drawn independently from a known distribution  $F$ . When bidder  $i$  arrives with valuation  $v_i$ , we offer it a price  $p_i$  that can depend on the distribution  $F$  and on  $i$  but not on  $v_i$ . If  $v_i \geq p_i$ , bidder  $i$  purchases the item and the process ends with revenue  $p_i$ . Otherwise the process continues with the next bidder if there is one. It is a natural question how to optimally set  $p_i$  for  $1 \leq i \leq n$  in order to maximize the expected revenue, and how large this expected revenue will be. The question we ask here is how much revenue can be obtained with *discriminatory* prices that may differ among bidders, compared to a single *anonymous* price  $p = p_1 = \dots = p_n$  offered to each of the bidders. Denoting by  $R_n^d$  and  $R_n^a$  the maximum revenue that can be obtained using discriminatory and anonymous pricing when there are  $n$  bidders, we will specifically be interested in bounds on the revenue gap  $R_n^d/R_n^a$ , and these bounds will generally depend on properties of the distribution  $F$ . Obviously  $R_n^d \geq R_n^a$  and thus  $R_n^d/R_n^a \geq 1$ .

Given the assumption that valuations are drawn from identical distributions, it obviously only makes sense to discriminate among bidders temporally and not based on their types. Regarding this assumption, two observations are worth making. First, identical distributions can arise not only when bidders actually have the same type, but also when discrimination based on type is infeasible or illegal.<sup>1</sup> Second, a model with identical distributions can then serve as an approximation to a setting where bidders have different types. For example,  $n$  independent draws from the empirical distribution over  $m$  valuations  $v_1, \dots, v_m$  approximate  $n$  values drawn sequentially without replacement given that  $n \ll m$ , but the former is much easier to analyze.<sup>2</sup>

A good way to understand optimal discriminatory and anonymous prices is via a framework developed by Myerson [19] to characterize revenue-optimal single-item auctions. An auction in this context is any mechanism that solicits bids, produces an assignment of the item and a price, and that is truthful in the sense that a bidder cannot increase its expected utility by submitting a bid that differs from its valuation. Myerson observed that the optimal auction problem can be written as a mathematical program subject to the constraint that the item is sold with probability at most one and subject to truthfulness constraints. Sequential posted prices clearly satisfy the constraints, so the optimal auction revenue provides an upper bound on  $R_n^d$  and  $R_n^a$ .

When there is a single bidder with valuation drawn from a distribution with density function  $f$  and cumulative distribution function  $F$ , the optimal auction and the optimal sequential mechanisms using discriminatory and anonymous prices are all the same and choose a monopoly price, i.e., a price that maximizes  $p(1 - F(p))$ . Analysis of the first-order conditions leads to a virtual value function  $\varphi$  given by  $\varphi(p) = p - (1 - F(p))/f(p)$ . For distributions that are regular in the sense that  $\varphi$  is strictly monotone,  $\varphi$  has an inverse and the monopoly price can be obtained as  $\varphi^{-1}(0)$ . The respective optimal mechanisms from the different classes do not usually coincide when there is more than one bidder, but virtual values turn out to be very useful in analyzing mechanisms from either class. They can be leveraged even for irregular distributions at the possible expense of producing lotteries over prices rather than prices, but obviously cannot be used directly for distributions that do not have a density.

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<sup>1</sup>In the US, the Robinson-Patman Act of 1936 for example prohibits anti-competitive practices such as price discrimination that favors chain stores over small retailers. Similar laws are in place in the EU.

<sup>2</sup>This line of reasoning was in fact applied by Moser [18] in analyzing a problem originally proposed by Cayley [6]. We will see more of this problem later.

Returning to sequential posted prices, it is not difficult to see that the optimal anonymous price is the monopoly price for the distribution of  $\max_i v_i$ , while the optimal discriminatory prices can be written recursively such that  $p_i$  is the monopoly price for the conditional distribution of  $v_i$  given that it exceeds the expected revenue from bidders  $j > i$  when using prices  $p_j$ . This explicit formulation of the optimal prices will be used in Section 5 to bound the revenue gap for a class of distributions that give rise to affine virtual value functions. We start, however, by deriving tight bounds for general distributions and for distributions that are regular. We do so not via virtual values but via a mathematical program that starts from the discriminatory pricing problem and relaxes the requirement of selling the item with probability at most one to hold only in expectation. Lower bounds are shown by giving distributions with the desired properties along with the optimal discriminatory and anonymous prices.

## 1.1 Our Contribution

While the exact question we study here has to our knowledge not previously been considered, a number of bounds for our setting can be obtained rather directly from results on related problems. We defer a detailed discussion of the relevant results and techniques but mention here upper and lower bounds of 2 for arbitrary distributions, respectively implied by work of Chawla et al. [8] and Chawla et al. [7], as well as an upper bound of  $e/(e-1)$  for regular distributions implied by the work of Chawla et al. [8]. These bounds are asymptotically equal to ours but do not hold for distributions without a density and may require lotteries over anonymous prices rather than a single anonymous price for distributions that are irregular.

For arbitrary distributions we obtain a bound of  $2 - 1/n$  that is tight for every value of  $n$ . The upper bound is shown via a mathematical program expressing the ratio between the maximum revenue from the ex-ante relaxation and that from anonymous prices. It thus avoids the limitations of techniques that apply prophet inequalities to virtual values. With some additional work the proof of the upper bound also leads to a matching lower bound, which turns out to use a slight modification of the distribution given by Chawla et al. [7] to show an asymptotic lower bound of 2. When  $F$  is regular, an improved upper bound of  $1/(1 - (1 - 1/n)^n)$  can be obtained by observing that the ex-ante relaxation has an optimal solution which sells the item with equal probability to all bidders. This bound is increasing in  $n$  and approaches  $e/(e-1) \approx 1.582$  as  $n$  goes to infinity. We again show that the bound is tight for every  $n$ , improving on a known lower bound of approximately 1.11.

From a technical perspective, the most interesting aspect of these results is a novel use of the ex-ante relaxation that also leads to new structural insights regarding the prices. The key insight is that a careful randomization over the prices achieving the optimum of the relaxed problem, when posted sequentially, approximates the revenue of the relaxed problem. The same can then be shown to hold for one of the prices, and in fact for one of the optimal discriminatory prices.

We finally consider distributions  $F$  for which the virtual value function is affine and show that this property leads to an interesting equivalence between revenue maximization and welfare maximization, where the welfare achieved by a given set of prices is the expected valuation  $v_i$  of the bidder  $i$  the item is sold to. The optimal price  $p_i$  for either objective is easily seen to depend only on the number of bidders remaining after  $i$ , and it turns out that the revenue-maximizing price with  $k$  remaining bidders is equal to the welfare-maximizing price with  $k+1$  remaining bidders. This result is useful because affinity of the virtual value function turns out to hold for generalized Pareto distributions, which include uniform, exponential, and Pareto distributions as special cases,

and because the welfare maximization problem is the well-studied Cayley-Moser problem from optimal stopping theory. We thus leverage existing results to obtain rather precise bounds of approximately 1.037 and 1.073 for the uniform and the exponential distribution.

## 1.2 Related Work

A question related to ours was first considered by Chawla et al. [7], who compared the maximum revenue from an auction to that of sequential posted prices in settings where virtual valuations are defined. They gave discriminatory prices approximating the optimal auction revenue within a factor of 3. For identical regular distributions these prices are in fact anonymous and an improved upper bound of 2.17 can be obtained. For identical irregular distributions the authors gave a lower bound of 2. It is not difficult to see that both the upper bound of 2.17 and the lower bound of 2 apply to the gap between discriminatory and anonymous pricing.

For the same problem and under the same assumptions, Chawla et al. [8] obtained tighter bounds by analyzing virtual values using results from optimal stopping theory, in particular the classic prophet inequality of Krengel and Sucheston [16, 17] and an improved version due to the authors for a setting where the bidders can be considered in a particular order. When specialized to identical distributions, these results imply upper bounds on the gap between discriminatory and anonymous pricing of 2 for irregular distributions and  $e/(e - 1)$  for regular distributions.

The best bounds for non-identical distributions are due to Alaei et al. [2], who formulated the revenue gap between the optimal auction and anonymous pricing as a mathematical program and through a series of relaxations showed an upper bound of  $e \approx 2.718$  for regular distributions. A major advantage of the direct analysis of the mathematical programs, like in our case, is that it applies to distributions without a density. The bound of  $e$  is tight for the gap between the relaxation of the mathematical program and anonymous pricing, for the gap between optimal auction revenue and anonymous pricing the best lower bound is approximately 2.23. For general distributions there is a tight bound of  $n$  even on the gap between discriminatory and anonymous pricing.

Blumrosen and Holenstein [4] gave a characterization of the optimal discriminatory and anonymous prices for the case of identical distributions, as well as formulas for the asymptotic revenue in terms of parameters of the distribution. For the regular power-law distribution they showed that the optimal revenue obtained by an auction, via discriminatory pricing, and through anonymous pricing is  $0.89\sqrt{n}$ ,  $0.71\sqrt{n}$ , and  $0.64\sqrt{n}$ , which implies an asymptotic lower bound of approximately  $0.71/0.64 \approx 1.11$  for the gap we are interested in.

The ex-ante relaxation was identified as a quantity of interest by Chawla et al. [7] and used subsequently for example by Yan [22] and Alaei [1]. The use of prophet inequalities in mechanism design was pioneered by Hajiaghayi et al. [13] and developed further in a series of papers [8, 15, 3, 1, 10, 21]. The variant of the secretary problem related to our problem, the Cayley-Moser problem, was proposed by Moser [18], solutions for the uniform and the exponential distribution were respectively given by Gilbert and Mosteller [12] and by Karlin [14]. The equivalence for distributions with affine virtual values between revenue maximization and welfare maximization with an additional bidder is somewhat reminiscent of a result of Bulow and Klemperer [5], which bounds the revenue of the optimal auction by the revenue of a second-price auction with an additional bidder. More distantly related to our work is work of Chen et al. [9], who analyzed a mathematical program to characterize optimal auctions with unlimited supply, and of Feldman et al. [11], who used anonymous posted prices in the design of mechanisms for combinatorial auctions with high welfare.

## 2 Preliminaries

Consider a set  $[n] = \{1, \dots, n\}$  of bidders with valuations  $v_1, \dots, v_n$  for a single item that is for sale, distributed independently according to a non-negative and possibly non-continuous probability distribution with cumulative distribution function  $F$ . Denote by  $\bar{F}(x)$  the left-sided limit of  $F$  at  $x$ , i.e.,  $\bar{F}(x) = \lim_{y \uparrow x} F(y)$ . Let the *revenue curve* of  $F$  be the function  $R : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$  such that  $R(q)$  is the expected revenue when selling an item to one of the bidders with probability exactly  $q$ , i.e.,  $R(q) = q\bar{F}^{-1}(1-q)$ , where, with a slight abuse of notation,  $\bar{F}^{-1}(1-q) = \max\{p \geq 0 : \bar{F}(p) \leq 1-q\}$ . By construction  $\bar{F}$  is left-continuous, so that the maximum is attained. Distribution  $F$  is called *regular* if  $R$  is concave.

A posted-price mechanism considers the bidders in turn, offers each of them a price, and sells the item to the first bidder whose valuation exceeds the price. The revenue of a mechanism then is the expected price at which the item is sold, taken over the joint distribution of bidders' valuations. A mechanism with *anonymous prices* offers the same price  $p$  to all of the bidders and obtains revenue

$$R_n^a(p) = p \left(1 - (\bar{F}(p))^n\right).$$

A mechanism with *discriminatory prices* uses a vector  $\mathbf{p} = (p_1, \dots, p_n)$  of prices, offering price  $p_i$  to bidder  $i$ , and obtains revenue

$$R_n^d(\mathbf{p}) = \sum_{i \in [n]} p_i \cdot (1 - \bar{F}(p_i)) \cdot \prod_{j \in [i-1]} \bar{F}(p_j).$$

We denote by  $R_n^a$  and  $R_n^d$  the maximum revenue from anonymous and discriminatory prices, i.e.,  $R_n^a = \max_{p \geq 0} R_n^a(p)$  and  $R_n^d = \max_{\mathbf{p} \in \mathbb{R}_{\geq 0}^n} R_n^d(\mathbf{p})$ . Clearly, for all  $n$ ,  $R_n^a \leq R_n^d$ .

The revenue from discriminatory prices is rather difficult to analyze directly, and it will sometimes be useful to consider instead an *ex-ante relaxation* of the associated maximization problem with revenue

$$R_n^x = \max \left\{ \sum_{i \in [n]} R(q_i) : \mathbf{q} \in [0, 1]^n, \sum_{i \in [n]} q_i \leq 1 \right\}.$$

Intuitively this quantity drops the requirement that there exists only one item and instead ensures that an item is sold with probability at most one in expectation over the random draws from  $F$ . The following result, proved in Appendix A, shows that the ex-ante relaxation indeed relaxes the discriminatory pricing problem.

**Lemma 1.** *For any distribution  $F$  and any  $n \in \mathbb{N}$ ,  $R_n^d \leq R_n^x$ .*

It is worth noting that  $R_n^x$ , like  $R_n^a$  and  $R_n^d$ , is defined in terms of the revenue curve  $R(q)$  and thus considers deterministic mechanisms rather than mechanisms that may offer lotteries over prices. For ease of exposition we assume all three quantities to exist and to be finite. All results can be seen to hold in general by standard limit arguments.

### 3 General Distributions

We begin by deriving an upper bound on the revenue gap for arbitrary distributions, allowing in particular for discontinuity and irregularity. The key insight here will be that a carefully chosen lottery over anonymous prices, corresponding to the probabilities of sale in an optimal solution to the ex-ante relaxation, provides a good approximation to the optimal ex-ante revenue. This is made precise by the following lemma.

**Lemma 2.** *For any number  $n$  of bidders with values drawn independently from an arbitrary distribution  $F$ , there exists a price  $p$  with  $p = \bar{F}^{-1}(1 - q_i)$  for some  $i \in [n]$  such that*

$$\frac{R_n^x}{R_n^a(p)} \leq \sum_{i \in [n]: q_i > 0} \frac{q_i}{1 - (\bar{F}(\bar{F}^{-1}(1 - q_i)))^n},$$

where  $q_i$  is the quantile chosen for bidder  $i$  in an optimal solution of the ex-ante relaxation.

*Proof.* The statement trivially holds if  $q_i = 0$  for all  $i \in [n]$ . If  $q_i > 0$  for some  $i \in [n]$  we have that  $\bar{F}(\bar{F}^{-1}(1 - q_i)) = \bar{F}(\max\{p \geq 0 : \bar{F}(p) \leq 1 - q_i\}) \leq 1 - q_i < 1$  and thus  $1 - (\bar{F}(\bar{F}^{-1}(1 - q_i)))^n > 0$ . For each  $i \in [n]$  let  $p_i = \bar{F}^{-1}(1 - q_i)$  and

$$\alpha_i = \begin{cases} \frac{q_i}{1 - (\bar{F}(\bar{F}^{-1}(1 - q_i)))^n} & \text{if } q_i > 0 \\ 0 & \text{otherwise,} \end{cases}$$

and consider a lottery over the anonymous prices  $p_1, \dots, p_n$  where  $p_i$  is chosen with probability  $\alpha_i / \sum_{i \in [n]} \alpha_i$ . Let  $R$  be the expected revenue obtained from this lottery. Then

$$\begin{aligned} R &= \sum_{i \in [n]: q_i > 0} \frac{\alpha_i}{\sum_{j \in [n]} \alpha_j} p_i \left(1 - (\bar{F}(p_i))^n\right) \\ &= \sum_{i \in [n]: q_i > 0} \frac{\alpha_i}{\sum_{j \in [n]} \alpha_j} \bar{F}^{-1}(1 - q_i) \left(1 - (\bar{F}(\bar{F}^{-1}(1 - q_i)))^n\right) \\ &= \frac{1}{\sum_{j \in [n]} \alpha_j} \sum_{i \in [n]: q_i > 0} q_i \bar{F}^{-1}(1 - q_i) \\ &= \frac{R_n^x}{\sum_{j \in [n]: q_j > 0} \alpha_j}, \end{aligned}$$

and thus  $R_n^x/R = \sum_{j \in [n]: q_j > 0} \alpha_j$ . As the lottery chooses a single price that is then offered to all bidders, there must also exist a deterministic price  $p \in \{p_1, \dots, p_n\}$  providing at least the same revenue.  $\square$

The upper bound in this lemma depends only on the probabilities of sale in the ex-ante relaxation, maximizing it subject to feasibility for the ex-ante relaxation thus yields an upper bound that applies to any distribution.

**Theorem 3.** *For any number  $n$  of bidders with values drawn independently from an arbitrary distribution  $F$ ,*

$$\frac{R_n^x}{R_n^a} \leq 2 - \frac{1}{n}.$$

*Proof.* Let  $q_i$  be the probability with which the item is sold to bidder  $i$  in the ex-ante relaxation, such that in particular  $\sum_{i \in [n]} q_i \leq 1$ . For distributions that are not continuous  $F(F^{-1}(1 - q_i))$  may not be equal to  $1 - q_i$ , but since  $F$  is non-decreasing it is always the case that

$$\begin{aligned}\bar{F}(\bar{F}^{-1}(1 - q_i)) &= \bar{F}(\max\{p \geq 0 : \bar{F}(p) \leq 1 - q_i\}) \\ &= \max\{\bar{F}(p) \geq 0 : \bar{F}(p) \leq 1 - q_i\} \\ &\leq 1 - q_i.\end{aligned}$$

Thus

$$\begin{aligned}\frac{R_n^x}{R_n^a} &\leq \sup \left\{ \sum_{i \in [n]} \frac{x_i}{1 - y_i^n} : \mathbf{x}, \mathbf{y} \in (0, 1]^n, y_i \leq 1 - x_i \text{ for } i \in [n], \sum_{i \in [n]} x_i \leq 1 \right\} \\ &= \sup \left\{ \sum_{i \in [n]} \frac{x_i}{1 - (1 - x_i)^n} : \mathbf{x} \in (0, 1]^n, \sum_{i \in [n]} x_i \leq 1 \right\} \\ &= \sup \left\{ \sum_{i \in [n]} \frac{1}{\sum_{k=1}^n \binom{n}{k} (-x_i)^{k-1}} : \mathbf{x} \in (0, 1]^n, \sum_{i \in [n]} x_i \leq 1 \right\},\end{aligned}$$

where the inequality holds by Lemma 2 and by setting  $y_i = \bar{F}(\bar{F}^{-1}(1 - q_i))$ , the first equality because for  $i \in [n]$ ,  $x_i/(1 - y_i^n)$  is increasing in  $y_i$ , and the second equality because  $(1 - x_i)^n = \sum_{k=0}^n \binom{n}{k} (-x_i)^k$ . Consider now the function  $g : \{\mathbf{x} \in [0, 1]^n : \sum_{i \in [n]} x_i \leq 1\} \rightarrow \mathbb{R}$  given by  $g(\mathbf{x}) = \sum_{i \in [n]} (n + \sum_{k=2}^n \binom{n}{k} (-x_i)^{k-1})^{-1}$ , and let

$$\mathbf{x}^* = \arg \max \left\{ g(\mathbf{x}) : \mathbf{x} \in [0, 1]^n, \sum_{i \in [n]} x_i \leq 1 \right\}.$$

Since  $g$  is continuous on its domain  $\mathbf{x}^*$  exists, and since  $\lim_{x \downarrow 0} x^0 = 1$ ,

$$\frac{R_n^x}{R_n^a} \leq g(\mathbf{x}^*).$$

To complete the proof we show that  $g(\mathbf{x}^*) = 2 - 1/n$ . We do so by means of three lemmas, which we prove in Appendix C.

Let  $N' = \{i \in [n] : x_i^* \in (0, 1)\}$ . Since  $g$  is differentiable, the local optimality conditions for  $\mathbf{x}^*$  imply that  $d/dx_i g(\mathbf{x}^*) = d/dx_j g(\mathbf{x}^*)$  for all  $i, j \in N'$ . By Lemma 13,  $g$  is strictly concave and thus  $x_i^* = x_j^*$  for all  $i, j \in N'$ , i.e., there exists  $z \in [0, 1]$  such that  $x_i^* = z$  for all  $i \in N'$ . By observing that the contribution of bidder  $i$  to  $g(\mathbf{x}^*)$  is equal to  $1/n$  when  $x_i^* = 0$  and equal to  $\frac{z}{1 - (1 - z)^n}$  when  $x_i^* = z$ , we obtain that

$$g(\mathbf{x}^*) = \max_{k \in \{0, \dots, n\}} \max_{z \in [0, 1] : zk \leq 1} \frac{n - k}{n} + \frac{zk}{1 - (1 - z)^n}.$$

By Lemma 14,  $\frac{z}{1 - (1 - z)^n}$  is increasing in  $z$  for  $z \in [0, 1]$ , and thus

$$g(\mathbf{x}^*) = \max_{k \in \{0, \dots, n\}} \frac{n - k}{n} + \frac{1}{1 - (1 - \frac{1}{k})^n}.$$

By Lemma 15 the maximum is equal to  $2 - 1/n$ , and the claim follows.  $\square$

A closer inspection of the proof of Lemma 2 reveals that we can in fact strengthen Theorem 3 to a comparison between an arbitrary set of discriminatory prices and the best anonymous price among them. A proof of this result is given in Appendix D.

**Corollary 4.** *For any number  $n$  of bidders with values drawn independently from an arbitrary distribution  $F$ , and any vector  $\mathbf{p} = (p_1, \dots, p_n)$  of prices, there exists  $p \in \{p_1, \dots, p_n\}$  such that*

$$\frac{R_n^d(\mathbf{p})}{R_n^a(p)} \leq 2 - \frac{1}{n}.$$

By looking at the proof of the upper bound one more time, we also obtain a matching lower bound that in fact applies to the gap  $R_n^d/R_n^a$  between discriminatory and anonymous pricing. Tightness of the upper bound in particular requires tightness of Lemma 15, which holds only when  $k = 1$ . Our goal will thus be to find a distribution for which the optimal discriminatory prices sell with vanishing probability to  $n - k = n - 1$  of the bidders while still extracting revenue  $1/n$  from each of them in expectation, and with probability and expected revenue approaching 1 to the remaining bidder. The distribution is irregular and closely related to one used by Chawla et al. [7] to establish an asymptotic lower bound of 2. We defer the proof to Appendix E.

**Theorem 5.** *For any  $\delta > 0$ , there exists a distribution  $F$  such that for  $n$  bidders with values drawn independently from  $F$ ,*

$$\frac{R_n^d}{R_n^a} \geq 2 - \frac{1}{n} - \delta.$$

## 4 Regular Distributions

The lower bound of Theorem 5 leaves open the possibility that a better upper bound exists for regular distributions. We indeed obtain such a bound, which again applies to the gap between the optimal anonymous pricing and the ex-ante relaxation and holds for every value of  $n$ . The bound approaches  $e/(e - 1) \approx 1.582$  as  $n$  grows and is thus asymptotically equal to a bound implied by a result of Chawla et al. [8]. It follows in a straightforward way from the observation that concavity of the revenue curve together with the constraint to sell the item with probability at most one causes the ex-ante relaxation to have a symmetric optimum.

**Theorem 6.** *For any number  $n$  of bidders with values drawn independently from a regular distribution  $F$ ,*

$$\frac{R_n^x}{R_n^a} \leq \frac{1}{1 - (1 - \frac{1}{n})^n}.$$

*Proof.* It is a standard exercise to show that the ex-ante relaxation has a symmetric optimum at  $q_i = q^* \leq 1/n$  for all bidders  $i \in [n]$ , a formal proof is included in Appendix F for completeness. Letting  $p^* = \bar{F}^{-1}(1 - q^*)$ , we have that  $R_n^x = nq^*\bar{F}^{-1}(1 - q^*) = nq^*p^*$ . By definition  $p^* = \max\{p \geq 0 : \bar{F}(p) \leq 1 - q^*\}$ , so,  $\bar{F}(p^*) \leq 1 - q^*$ . So setting  $p^*$  as an anonymous price shows that  $R_n^a \geq (1 - (F(p^*))^n)p^* \geq (1 - (1 - q^*)^n)p^*$ . Thus

$$\frac{R_n^x}{R_n^a} \leq \frac{nq^*p^*}{(1 - (1 - q^*)^n)p^*} = \frac{nq^*}{1 - (1 - q^*)^n} \leq \frac{1}{1 - (1 - \frac{1}{n})^n}.$$

The second inequality holds because  $q^* \leq 1/n$  and, by Lemma 14,  $\frac{q^*}{1 - (1 - q^*)^n}$  is increasing in  $q^*$ .  $\square$



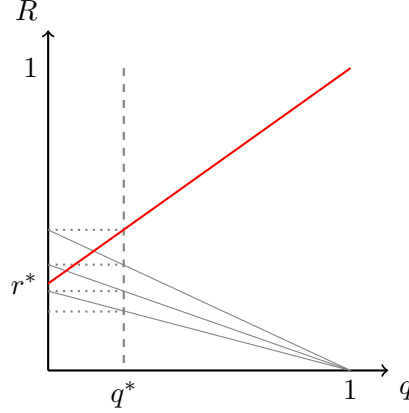


Figure 1: Limiting revenue curve providing a tight lower bound for regular distributions, shown for the case where  $n = 4$ . The optimal discriminatory prices sell with probability approaching 0 to each of the first  $n - 1$  bidders and with probability approaching 1 to the last bidder. For an appropriate choice of  $r^*$  the optimal anonymous price sells with probability  $q^* = 1/n$  to each bidder assuming that the item has not yet been sold, and thus obtains a revenue of  $(r^* + q^*(1 - r^*))(1 + (1 - q^*) + (1 - q^*)^2 + (1 - q^*)^3)$ . This latter revenue corresponds to the values of  $R$  where the dashed vertical line at  $q^*$  intersects the revenue curve and the three solid gray lines descending from the value of  $R$  for the respective previous bidder.

It is interesting to note that this upper bound is ultimately obtained in the same way as that of Theorem 3, by canceling out the maximum revenue from the ex-ante distribution. Whereas here cancellation takes place explicitly and more or less automatically because the optimum of the ex-ante relaxation is symmetric, the proof of Theorem 3 and of Lemma 2 in particular required careful balancing of the contributions of the different bidders.

Analogously to Corollary 4, we also obtain the following.

**Corollary 7.** *For any number  $n$  of bidders with values drawn independently from an arbitrary regular distribution  $F$ , and any vector  $\mathbf{p} = (p_1, \dots, p_n)$  of prices, there exists  $p \in \{p_1, \dots, p_n\}$  such that*

$$\frac{R_n^d(\mathbf{p})}{R_n^a(p)} \leq \frac{1}{1 - (1 - \frac{1}{n})^n}.$$

Obtaining a good lower bound under the additional assumption of regularity turns out to be rather more challenging. To this end we consider a sequence of distributions with a limiting revenue curve  $R$  that is linear and satisfies  $R(1) = 1$  and  $\lim_{q \downarrow 0} R(q) = r \in [0, 1]$ . Discriminatory pricing can sell the item with probability approaching 0 to each of the first  $n - 1$  bidders and with probability approaching 1 to the last bidder, thus obtaining a revenue of almost  $(n - 1)r + 1$ . Anonymous pricing can set a price that for the first bidder sells with probability  $q \in [0, 1]$  and obtains a revenue of  $r + q(1 - r)$ . Revenue from each consecutive bidder will be smaller by an additional factor of  $(1 - q)$ , leading to an overall revenue of  $(r + q(1 - r)) \sum_{i=0}^{n-1} (1 - q)^i$ . The situation is illustrated in Figure 1. The difficulty will now be to choose  $r$  such that for the corresponding optimal choice of  $q$  the gap between discriminatory and anonymous pricing is as large as possible. It is worth noting that the eventual choice of  $r$  will lead to  $q = 1/n$ , which by the upper bound of Theorem 6 and by the existence of a symmetric optimum of the ex-ante relaxation is necessary for tightness. We thus obtain the following result, which is tight for every value of  $n$  and proved formally in Appendix G.

**Theorem 8.** *For any  $\delta > 0$ , there exists a regular distribution  $F$  such that for  $n$  bidders with values drawn independently from  $F$ ,*

$$\frac{R_n^d}{R_n^a} \geq \frac{1}{1 - (1 - \frac{1}{n})^n} - \delta.$$

## 5 Distributions with Affine Virtual Values

We now turn to distributions  $F$  with the property that the virtual value function  $\varphi(x) = x - (1 - F(x))/f(x)$  exists and is an affine function, such that  $\varphi(x) = ax + b$  for some  $a, b \in \mathbb{R}$ . We will see that for this class of distributions, finding discriminatory prices that maximize revenue is equivalent to finding discriminatory prices that maximize welfare when there is one additional bidder. Here, the welfare achieved by a given set of prices is the valuation  $v_i$  of the bidder  $i$  the item is sold to rather than the price at which the item is sold.

We have seen that the discriminatory price offered to a particular bidder to maximize revenue only depends on the number of additional bidders still remaining, and the same is true for welfare maximization as well. Fix a distribution with affine virtual values. For  $i \in \mathbb{N}$ , let  $R_i$  and  $W_i$  respectively denote the optimal revenue and optimal welfare obtainable from  $i$  bidders by discriminatory pricing. The price  $p_i^R$  offered to maximize when an additional  $i$  bidders remain after the current one must satisfy

$$p_i^R \in \arg \max_{p \geq 0} \left( p(1 - F(p)) + F(p)R_i \right).$$

First-order optimality conditions imply that  $(1 - F(p)) - pf(p) + f(p)R_i = 0$ , or equivalently that

$$\varphi(p_i^R) = R_i. \quad (1)$$

Analogously, when maximizing welfare, the price  $p_i^W$  must satisfy

$$p_i^W \in \arg \max_{p \geq 0} \left( \mathbb{E}_{X \sim F}[X \mid X \geq p](1 - F(p)) + F(p)W_i \right) = \arg \max_{p \geq 0} \left( \int_p^\infty xf(x) dx + F(p)W_i \right),$$

and first-order conditions imply that

$$p_i^W = W_i. \quad (2)$$

The equivalence between revenue maximization and welfare maximization with one additional bidder can now be formulated as follows.

**Theorem 9.** *For any distribution  $F$  with affine virtual values, and any  $i \in \mathbb{N}$ ,  $p_i^R = p_{i+1}^W$ .*

*Proof.* We begin by recalling a basic property of virtual values, namely that for all  $y \geq 0$ ,

$$\mathbb{E}_{X \sim F}[\varphi(X) \mid X \geq y] = y. \quad (3)$$

Indeed,

$$\begin{aligned} \mathbb{E}_{X \sim F}[\varphi(X) \mid X \geq y] &= \int_y^\infty \left( x - \frac{1 - F(x)}{f(x)} \right) f(x) dx \\ &= \left[ xF(x) \right]_y^\infty - \int_y^\infty F(x) dx - \int_y^\infty 1 - F(x) dx \\ &= \left[ x(F(x) - 1) \right]_y^\infty = y. \end{aligned}$$

We will now prove the claim by induction on  $i$ . For  $i = 0$ ,

$$\varphi(\mathbb{E}_{X \sim F}[X]) = \mathbb{E}_{X \sim F}[\varphi(X)] = \mathbb{E}_{X \sim F}[\varphi(X) \mid X \geq 0] = 0,$$

where the first and last equality respectively hold by linearity of  $\varphi$  and by (3). This implies that  $p_0^R = \varphi^{-1}(0) = \mathbb{E}_{X \sim F}[X] = p_1^W$ . For  $i \geq 1$ , we claim that

$$\begin{aligned} p_{i+1}^W &= W_{i+1} \\ &= F(p_i^W)W_i + (1 - F(p_i^W))\mathbb{E}_{X \sim F}[X \mid X \geq p_i^W] \\ &= F(p_i^W)W_i + (1 - F(p_i^W))\varphi^{-1}(p_i^W) \\ &= F(p_{i-1}^R)p_{i-1}^R + (1 - F(p_{i-1}^R))\varphi^{-1}(p_{i-1}^R) \\ &= F(p_{i-1}^R)\varphi^{-1}(R_{i-1}) + (1 - F(p_{i-1}^R))\varphi^{-1}(p_{i-1}^R) \\ &= \varphi^{-1}(F(p_{i-1}^R)R_{i-1} + (1 - F(p_{i-1}^R))p_{i-1}^R) \\ &= \varphi^{-1}(R_i) = p_i^R. \end{aligned}$$

The first equality holds by (2), the second uses that  $W_{i+1} = W_i$  if the value of the current bidder is below  $p_i^W$  and  $W_{i+1} = \mathbb{E}_{X \sim F}[X \mid X \geq p_i^W]$  otherwise. The third equality holds by (3), the fourth because by (2) and the induction hypothesis we have that  $W_i = p_i^W = p_{i-1}^R$ . The fifth equality finally holds by (1), the sixth by linearity of  $\varphi$ , and the seventh equality finally uses that  $R_i = R_{i-1}$  if value of the current bidder is below  $p_{i-1}^R$  and  $R_i = p_i^R$  otherwise.  $\square$

This result is interesting not only from a structural perspective, and the reason is twofold. First, affinity of virtual values characterizes an interesting class of distributions that includes the uniform and the exponential distribution. Second, the welfare-maximization problem has been studied extensively in the area of optimal stopping theory, where it is known as the Cayley-Moser problem, so that existing explicit or approximate solutions for the optimal prices can be used to bound the maximum revenue from discriminatory pricing.

Call distribution  $F$  a generalized Pareto distribution if there exist  $\mu, \lambda, \xi$  with  $\lambda > 0$  and  $\xi \geq 0$  such that

$$F(x) = \begin{cases} 1 - (1 - \xi\lambda(x - \mu))^{\frac{1}{\xi}} & \text{if } \xi > 0 \\ 1 - e^{\lambda(x - \mu)} & \text{if } \xi = 0 \end{cases}$$

for all  $x \in [\mu, \mu + 1/(\xi\lambda)]$  if  $\xi > 0$  and all  $x \geq \mu$  if  $\xi = 0$ . Clearly  $F$  is the uniform distribution on  $[\mu, \mu + 1/\lambda]$  if  $\xi = 1$  and the exponential distribution with parameter  $\lambda$  if  $\xi = \mu = 0$ . We have the following.

**Lemma 10** (Niazadeh et al. [20]). *The virtual value function  $\varphi(x) = x - (1 - F(x))/f(x)$  is affine if and only if  $F$  is a generalized Pareto distribution.*

Rather accurate bounds on the optimal prices in the Cayley-Moser problem are known for example for uniform distributions [12] and exponential distributions [14]. These can now be used to derive bounds for our setting.

**Theorem 11.** *For any number  $n$  of bidders with values drawn independently from a uniform distribution,*

$$\frac{R_n^d}{R_n^a} \leq \frac{R_{11}^d}{R_{11}^a} \approx 1.0368.$$

Similar calculations for exponential distributions show that the maximum of the ratio  $R_n^d/R_n^a$  is attained at  $n = 213$ , where it is roughly 1.0732.

## A Proof of Lemma 1

We claim that

$$\begin{aligned}
R_n^d &= \max \left\{ \sum_{i \in [n]} p_i (1 - \bar{F}(p_i)) \prod_{j \in [i-1]} \bar{F}(p_j) : \mathbf{p} \in \mathbb{R}_{\geq 0}^n \right\} \\
&= \max \left\{ \sum_{i \in [n]} p_i q_i : \mathbf{p} \in \mathbb{R}_{\geq 0}^n, q_i = (1 - \bar{F}(p_i)) \prod_{j \in [i-1]} \bar{F}(p_j) \text{ for all } i \in [n] \right\} \\
&\leq \max \left\{ \sum_{i \in [n]} p_i q_i : \mathbf{p} \in \mathbb{R}_{\geq 0}^n, \mathbf{q} \in [0, 1]^n, q_i \leq (1 - \bar{F}(p_i)) \text{ for all } i \in [n], \sum_{i \in [n]} q_i \leq 1 \right\} \\
&\leq \max \left\{ \sum_{i \in [n]} \bar{F}^{-1}(1 - q_i) q_i : \mathbf{q} \in [0, 1]^n \text{ for all } i \in [n], \sum_{i \in [n]} q_i \leq 1 \right\} = R_n^x.
\end{aligned}$$

The first equality rewrites the optimization problem in terms of  $q_i$ . The first inequality holds because  $q_i$  on the left-hand side satisfies the constraints on the right-hand side. The second inequality holds because  $q_i \leq 1 - \bar{F}(p_i)$  and thus  $\bar{F}(p_i) \leq 1 - q_i$ , and  $\bar{F}^{-1}(1 - q_i)$  is the largest such price. The final equality holds by definition.

## B An Auxiliary Lemma

The following result is a variant of Bernoulli's inequality that we will require in some of the proofs below.

**Lemma 12.** *Let  $k \in \mathbb{N}$  and  $x \in (0, 1)$ . Then  $(1 - x)^k < (1 + kx)^{-1}$ .*

*Proof.* We show the statement by induction. For  $k = 1$  the claim is that  $1 - x < (1 + x)^{-1}$ , which holds when  $x > 0$ . Then,

$$\begin{aligned}
(1 - x)^{k+1} &= (1 - x)^k (1 - x) \\
&< (1 + kx)^{-1} (1 + x)^{-1} \\
&= (1 + kx + x + kx^2)^{-1} \\
&\leq (1 + (k + 1)x)^{-1},
\end{aligned}$$

where the first inequality uses the induction hypothesis, the second inequality that  $rx^2 \geq 0$ , and both inequalities that  $x \geq 0$  and  $1 - x \geq 0$ .  $\square$

## C Lemmas Used in the Proof of Theorem 3

**Lemma 13.** *The function  $g(\mathbf{x}) = \sum_{i \in [n]} \frac{x_i}{1 - (1 - x_i)^n}$  is strictly concave on  $(0, 1)^n$ .*

*Proof.* The Hessian  $\nabla^2 g$  is a diagonal matrix where the  $i$ -th entry on the diagonal is equal to  $\frac{d^2}{dx_i^2} \left( \frac{x_i}{1 - (1 - x_i)^n} \right)$ . For any  $i \in \{1, \dots, n\}$  and  $x = x_i$ , we calculate

$$\frac{d^2}{dx^2} \left( \frac{x}{1 - (1 - x)^n} \right) = \frac{d}{dx} \left( \frac{1 - (1 - x)^n - nx(1 - x)^{n-1}}{(1 - (1 - x)^n)^2} \right)$$

$$\begin{aligned}
&= \frac{(n(1-x)^{n-1} - n(1-x)^{n-1} + nx(n-1)(1-x)^{n-2})(1-(1-x)^n)^2}{(1-(1-x)^n)^4} \\
&\quad - \frac{(1-(1-x)^n - nx(1-x)^{n-1})(2n(1-(1-x)^n)(1-x)^{n-1})}{(1-(1-x)^n)^4} \\
&= \frac{nx(n-1)(1-x)^{n-2}(1-(1-x)^n)}{(1-(1-x)^n)^3} - \frac{(1-(1-x)^n - nx(1-x)^{n-1})2n(1-x)^{n-1}}{(1-(1-x)^n)^3} \\
&= \frac{n(1-x)^{n-2}}{(1-(1-x)^n)^3} \left( nx - x - nx(1-x)^n + x(1-x)^n \right. \\
&\quad \left. - 2 + 2x + 2(1-x)^n - 2x(1-x)^n + 2nx(1-x)^n \right) \\
&= \frac{n(1-x)^{n-2}}{(1-(1-x)^n)^3} ((1-x)^n(nx - x + 2) + nx + x - 2).
\end{aligned}$$

We claim that this expression is strictly positive when  $x \in (0, 1)$ . Since  $n(1-x)^{n-2} > 0$  and  $(1-(1-x)^n)^3 > 0$ , it suffices to show that  $(1-x)^n(nx - x + 2) + nx + x - 2 > 0$ . To this end, note that for  $x = 0$  we have  $(1-x)^n(nx - x + 2) + nx + x - 2 = 0$ . On the other hand, for  $x \in (0, 1)$ ,

$$\begin{aligned}
&\frac{d}{dx}((1-x)^n(nx - x + 2) + nx + x - 2) \\
&= -n(1-x)^{n-1}(nx - x + 2) + (1-x)^n(n-1) + n + 1 \\
&= -n(1-x)^{n-1}(nx - x + 1) - (1-x)^{n-1}n + (1-x)^{n-1}(n - nx + x - 1) + n + 1 \\
&= -n(1-x)^{n-1}(nx - x + 1) - (1-x)^{n-1}n - (1-x)^{n-1}(nx - x + 1) + (1-x)^{n-1}n + n + 1 \\
&= (n+1)(1-(1-x)^{n-1}(nx - x + 1)) > 0.
\end{aligned}$$

Here the inequality holds because  $n+1 > 0$  and because, by Lemma 12,

$$(1-x)^{n-1}(nx - x + 1) < (1 + (n-1)x)^{-1}(nx - x + 1) = 1. \quad \square$$

**Lemma 14.** *The function  $\frac{z}{1-(1-z)^n}$  is increasing in  $z$  for  $z \in [0, 1]$ .*

*Proof.* By Lemma 13

$$\frac{d^2}{dz^2} \left( \frac{z}{1-(1-z)^n} \right) > 0$$

for  $z \in (0, 1)$ , so it suffices to show that

$$\lim_{z \downarrow 0} \frac{d}{dz} \left( \frac{z}{1-(1-z)^n} \right) \geq 0.$$

Indeed,

$$\begin{aligned}
\lim_{z \downarrow 0} \frac{d}{dz} \left( \frac{z}{1-(1-z)^n} \right) &= \lim_{z \downarrow 0} \frac{1 - (1-z)^n - nz(1-z)^{n-1}}{(1-(1-z)^n)^2} \\
&= \lim_{z \downarrow 0} \frac{n(1-z)^{n-1} - n(1-z)^{n-1} + nz(n-1)(1-z)^{n-2}}{2(1-(1-z)^n)n(1-z)^{n-1}}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{z \downarrow 0} \frac{n(n-1)z}{2n(1-z)(1-(1-z)^n)} \\
&= \lim_{z \downarrow 0} \frac{n(n-1)}{2n(-1 + (1-z)^n + n(1-z)^n)} = \frac{n-1}{2n},
\end{aligned}$$

where for the second and fourth equalities we have used l'Hospital's rule.  $\square$

**Lemma 15.**  $\max_{k \in \{1, \dots, n\}} \left( \frac{n-k}{n} + \frac{1}{1-(1-\frac{1}{k})^n} \right) = 2 - \frac{1}{n}$

*Proof.* We claim that

$$\begin{aligned}
\frac{n-k}{n} + \frac{1}{1-(1-\frac{1}{k})^n} &= \frac{n-k}{n} + \frac{1}{1-(\frac{k-1}{k})^n} \\
&= \frac{n-k}{n} + \frac{k^n}{k^n - (k-1)^n} \\
&= 1 + \frac{nk^n - k(k^n - (k-1)^n)}{n(k^n - (k-1)^n)} \\
&\leq 1 + \frac{(n-1)(k^n - (k-1)^n)}{n(k^n - (k-1)^n)} \\
&= 1 + \frac{n-1}{n} = 2 - \frac{1}{n},
\end{aligned}$$

where only the inequality is nontrivial. For the inequality it suffices to show that

$$nk^n - k^{n+1} + k(k-1)^n \leq (n-1)k^n - (n-1)(k-1)^n,$$

i.e., that

$$(n+k-1)(k-1)^n \leq k^n(k-1). \quad (4)$$

The latter clearly holds when  $k=1$ . When  $k>1$ , then by Bernoulli's inequality

$$1 + \frac{n-1}{k} \leq 1 + \frac{n-1}{k-1} \leq \left(1 + \frac{1}{k-1}\right)^{n-1}.$$

By taking logarithms on both sides and rearranging,

$$\log \frac{n+k-1}{k} \leq (n-1) \log \frac{k}{k-1},$$

$$\log(n+k-1) - \log k \leq (n-1)(\log k - \log(k-1)),$$

and

$$\log(n+k-1) + (n-1) \log(k-1) \leq n \log k.$$

By exponentiating both sides

$$(n+k-1)(k-1)^{n-1} \leq k^n,$$

and multiplying by  $(k-1)$  shows (4).  $\square$

## D Proof of Corollary 4

Consider a vector  $\mathbf{p} = (p_1, \dots, p_n)$  of prices, and a restriction  $R_n^x(\mathbf{p})$  of the ex-ante relaxation that is only allowed to charge price  $p_i$  to bidder  $i$ , i.e.,

$$R_n^x(\mathbf{p}) = \max \left\{ \sum_{i \in [n]} p_i q_i : \mathbf{q} \in [0, 1]^n, q_i \leq (1 - \bar{F}(p_i)) \text{ for all } i \in [n], \sum_{i \in [n]} q_i \leq 1 \right\}.$$

As an intermediate step of the proof of Lemma 1, we showed that  $R_n^x(\mathbf{p}) \geq R_n^d(\mathbf{p})$ . It is without loss of generality to assume that  $q_i > 0$  for some  $i \in [n]$ . We can then proceed with a calculation similar to that in the proof of Lemma 2, by considering a lottery over the prices  $p_1, \dots, p_n$  in which  $p_i$  is chosen with probability  $\alpha_i / \sum_{i \in [n]} \alpha_i$  with

$$\alpha_i = \begin{cases} \frac{q_i}{1 - \bar{F}(p_i)^n} & \text{if } q_i > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The expected revenue obtained from this lottery is equal to

$$R = \sum_{i \in [n]: q_i > 0} \frac{\alpha_i}{\sum_{j \in [n]} \alpha_j} p_i (1 - \bar{F}(p_i)^n) = \frac{R_n^x(\mathbf{p})}{\sum_{j \in [n]} \alpha_j}$$

which implies the existence of a price  $p \in \{p_1, \dots, p_n\}$  with

$$\frac{R_n^d(p_1, \dots, p_n)}{R^a(p)} \leq \sum_{i \in [n]} \frac{q_i}{1 - (\bar{F}(p_i))^n}. \quad (5)$$

This ratio is at most

$$\sup \left\{ \sum_{i \in [n]} \frac{x_i}{1 - y_i^n} : \mathbf{x}, \mathbf{y} \in [0, 1]^n, y_i \leq 1 - x_i \text{ for all } i \in [n], \sum_{i \in [n]} x_i \leq 1 \right\} \leq 2 - 1/n,$$

where the inequality follows by the same argument as in the proof of Theorem 3.

## E Proof of Theorem 5

Let  $\epsilon > 0$ , and consider the discrete distribution with support  $\{1, \frac{n}{\epsilon}\}$  and  $P[v_i = 1] = 1 - \frac{\epsilon}{n^2}$  and  $\Pr[v_i = \frac{n}{\epsilon}] = \frac{\epsilon}{n^2}$ . Clearly the optimal anonymous price  $p$  must be either 1 or  $\frac{n}{\epsilon}$ , and we have that  $R_n^a(1) = 1$  and  $R_n^a(\frac{n}{\epsilon}) = (1 - (1 - \frac{\epsilon}{n^2})^n) \frac{n}{\epsilon} \leq \sum_{i \in [n]} \frac{\epsilon}{n^2} \cdot \frac{n}{\epsilon} = 1$ . On the other hand, for discriminatory prices  $\mathbf{p} = (\frac{n}{\epsilon}, \dots, \frac{n}{\epsilon}, 1)$ , we have that  $\lim_{\epsilon \rightarrow 0} R_n^x(\mathbf{p}) = \lim_{\epsilon \rightarrow 0} (\sum_{i=1}^{n-1} (1 - \frac{\epsilon}{n^2})^{i-1} \frac{\epsilon}{n^2} \cdot \frac{n}{\epsilon} + (1 - \frac{\epsilon}{n^2})^{n-1} \cdot 1) = (n-1) \cdot \frac{1}{n} + 1 = 2 - \frac{1}{n}$ . Choosing  $\epsilon$  small enough yields the desired result.

## F Lemma Used in the Proof of Theorem 6

**Lemma 16.** *For any number  $n$  of bidders with values drawn from a regular distribution  $F$ , there exists  $q^* \leq 1/n$  such that the ex-ante relaxation has an optimum where  $q_i = q^*$  for all  $i \in [n]$ .*

*Proof.* The objective of the ex-ante relaxation is a sum of concave functions and therefore concave. Both the objective and the constraints are invariant under permutations of the set of bidders. Consider an optimal solution  $q^*$  of the ex-ante relaxation, and let  $\bar{q}^* = (1/n!) \sum_{\pi \in S_n} (P_\pi q^*)$ , where  $S_n$  is the set of permutations of  $[n]$  and  $P_\pi$  is the permutation matrix corresponding to permutation  $\pi$ . Then  $\bar{q}^*$  is feasible. Furthermore,  $\bar{q}^*$  is invariant under any permutation, so there must be some  $\alpha \in \mathbb{R}_{\geq 0}$  such that  $\bar{q}^* = \alpha \mathbf{1}$ . This implies in particular that  $\bar{q}^* = \alpha \mathbf{1}$  with  $\alpha \leq 1/n$ . Finally, by concavity of the objective and by Jensen's inequality,  $R(\bar{q}^*) \geq \frac{1}{n!} \sum_{\pi \in S_n} R(P_\pi q^*) = R(q^*)$ , so  $\bar{q}^*$  is optimal.  $\square$

## G Proof of Theorem 8

Let  $r \in \mathbb{R}$  and  $\epsilon > 0$ , and consider a piecewise linear revenue curve  $R$  with a slope  $r/\epsilon$  on  $(0, \epsilon)$  and slope  $\frac{1-r}{1-\epsilon}$  on  $(\epsilon, 1)$ , given by

$$R(q) = \begin{cases} \frac{r}{\epsilon} \cdot q & \text{for } q \leq \epsilon, \text{ and} \\ \frac{1-r}{1-\epsilon} \cdot q + (1 - \frac{1-r}{1-\epsilon}) & \text{for } q \geq \epsilon \end{cases}$$

Discriminatory prices of  $\bar{F}^{-1}(1 - \epsilon) = \frac{r}{\epsilon} \epsilon = r$  for the first  $n - 1$  bidders and  $\bar{F}^{-1}(0) = 1$  for the last bidder show that  $R_n^d \geq \sum_{i=1}^{n-1} (1 - \epsilon)^{i-1} \epsilon \frac{r}{\epsilon} + (1 - \epsilon)^{n-1} 1$ . On the other hand,  $R(q) \leq r + q(1 - r)$  for all  $q \in [0, 1]$ , so an anonymous price of  $\bar{F}^{-1}(1 - q)$  for  $q \in [0, 1]$  achieves revenue at most  $(r + q(1 - r)) \sum_{i=0}^{n-1} (1 - q)^i$ . For  $n \in [n]$  let

$$T_n = \lim_{\epsilon \rightarrow 0} \max_{r \in [0, 1]} \min_{q \in [0, 1]} \frac{\sum_{i=1}^{n-1} (1 - \epsilon)^{i-1} \epsilon \frac{r}{\epsilon} + (1 - \epsilon)^{n-1} 1}{(r + q(1 - r)) \sum_{i=0}^{n-1} (1 - q)^i},$$

and observe that it suffices to show that

$$T_n \geq \frac{1}{1 - (1 - \frac{1}{n})^n}.$$

First, note that for a continuous function  $f : X \times Y \rightarrow \mathbb{R}$  with  $X$  and  $Y$  compact, we have that  $\min_{x \in X} f(x, y)$  is continuous in  $y$ . Applying this argument two times, we conclude that  $\max_{r \in [0, 1]} \min_{q \in [0, 1]} \frac{\sum_{i=1}^{n-1} (1 - \epsilon)^{i-1} \epsilon \frac{r}{\epsilon} + (1 - \epsilon)^{n-1} 1}{(r + q(1 - r)) \sum_{i=0}^{n-1} (1 - q)^i}$  is continuous in  $\epsilon$  and we may exchange the order of the limit and the minimization and maximization and obtain

$$\begin{aligned} T_n &= \max_{r \in [0, 1]} \min_{q \in [0, 1]} \lim_{\epsilon \rightarrow 0} \frac{\sum_{i=1}^{n-1} (1 - \epsilon)^{i-1} \epsilon \frac{r}{\epsilon} + (1 - \epsilon)^{n-1} 1}{(r + q(1 - r)) \sum_{i=0}^{n-1} (1 - q)^i} \\ &= \max_{r \in [0, 1]} \min_{q \in [0, 1]} \frac{(n - 1)r + 1}{(r + q(1 - r)) \sum_{i=0}^{n-1} (1 - q)^i} \\ &= \max_{r \in [0, 1]} \min_{q \in [0, 1]} \frac{((n - 1)r + 1)q}{(q + r(1 - q))(1 - (1 - q)^n)} \end{aligned} \tag{6}$$

where the first equality holds by continuity of the maximum and minimum, the second equality by taking the limit, and the third equality because  $\sum_{i=0}^{n-1} (1 - q)^i q = (1 - (1 - q)^n)$ . The objective



is continuous in  $q$  for any  $r \in [0, 1]$ , so the minimum is attained. For a given  $r \in [0, 1]$  the latter happens if and only if

$$\frac{(q + r(1 - q))(1 - (1 - q)^n)}{q} \quad (7)$$

is maximized, and either at the boundary of  $[0, 1]$  or in its interior. We will assume the latter and show later that this assumption is valid for our eventual choice of  $r$ .

For (7) to be maximized at  $q \in (0, 1)$  it must be the case that

$$\begin{aligned} & \frac{d}{dq} \frac{(q + r(1 - q))(1 - (1 - q)^n)}{q} \\ &= \frac{[(1 - r)(1 - (1 - q)^n) + (q + r(1 - q))n(1 - q)^{n-1}]q - (q + r(1 - q))(1 - (1 - q)^n)}{q^2} \\ &= \frac{r[nq(1 - q)^n - q(1 - (1 - q)^n) - (1 - q)(1 - (1 - q)^n)]}{q^2} \\ & \quad + \frac{q(1 - (1 - q)^n) + nq^2(1 - q)^{n-1} - q(1 - (1 - q)^n)}{q^2} \\ &= \frac{r[(nq + 1)(1 - q)^n - 1] + nq^2(1 - q)^{n-1}}{q^2} = 0, \end{aligned}$$

i.e., that

$$r = \frac{nq^2(1 - q)^{n-1}}{1 - (nq + 1)(1 - q)^n}. \quad (8)$$

This equation cannot easily be solved for  $q$ , but it establishes a one-to-one relationship between values  $q \in (0, 1)$  and  $r \in (0, \frac{2}{n+1})$  because

$$\begin{aligned} \lim_{q \downarrow 0} \frac{nq^2(1 - q)^{n-1}}{1 - (nq + 1)(1 - q)^n} &= \lim_{q \downarrow 0} \frac{2nq(1 - q)^{n-1} - nq^2(n - 1)(1 - q)^{n-2}}{-n(1 - q)^n + (nq + 1)n(1 - q)^{n-1}} \\ &= \lim_{q \downarrow 0} \frac{nq(2 - 2q - nq + q)(1 - q)^{n-2}}{(-n + nq + n^2q + n)(1 - q)^{n-1}} \\ &= \lim_{q \downarrow 0} \frac{nq(2 - (n + 1)q)(1 - q)^{n-2}}{nq(n + 1)(1 - q)^{n-1}} \\ &= \lim_{q \downarrow 0} \frac{2 - (n + 1)q}{(n + 1)(1 - q)} = \frac{2}{n + 1}, \end{aligned}$$

where the first equality uses l'Hospital's rule,

$$\lim_{q \uparrow 1} \frac{nq^2(1 - q)^{n-1}}{1 - (nq + 1)(1 - q)^n} = 0,$$

and because for  $q \in (0, 1)$ ,

$$\begin{aligned} & \frac{d}{dq} \frac{nq^2(1 - q)^{n-1}}{1 - (nq + 1)(1 - q)^n} \\ &= \frac{nq(2 - (n + 1)q)(1 - q)^{n-2}(1 - (nq + 1)(1 - q)^n) - nq(n + 1)(1 - q)^{n-1}nq^2(1 - q)^{n-1}}{(1 - (nq + 1)(1 - q)^n)^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{nq(1-q)^{n-2}[(2-(n+1)q)(1-(nq+1)(1-q)^n) - (n+1)nq^2(1-q)^n]}{(1-(nq+1)(1-q)^n)^2} \\
&= \frac{nq(1-q)^{n-2}[2-(n+1)q - (1-q)^n(2nq+2-n^2q^2-nq^2-nq-q+n^2q^2+nq^2)]}{(1-(nq+1)(1-q)^n)^2} \\
&= \frac{nq(1-q)^{n-2}(2-(n+1)q - (2+(n-1)q)(1-q)^n)}{(1-(nq+1)(1-q)^n)^2} < 0.
\end{aligned}$$

For the inequality note that  $nq(1-q)^{n-2} > 0$  and, by Lemma 12,  $1-(nq+1)(1-q)^n > 0$ . It thus suffices to show that  $2-(n+1)q - (2+(n-1)q)(1-q)^n < 0$ . This is true because the left-hand side is equal to 0 when  $q = 0$  and because

$$\begin{aligned}
&\frac{d}{dq} 2-(n+1)q - (2+(n-1)q)(1-q)^n \\
&= -(n+1) - (n-1)(1-q)^n + (2+(n-1)q)n(1-q)^{n-1} \\
&= (1-q)^{n-1}(-n+nq+1-q+2n+n^2q-nq) - (n+1) \\
&= (1-q)^{n-1}(n^2q-q+n+1) - (n+1) \\
&= (1-q)^{n-1}(n+1)((n-1)q+1) - (n+1) \\
&= (n+1)((n-1)q+1)(1-q)^{n-1} - 1 < 0,
\end{aligned}$$

where the inequality holds because  $n+1 > 0$  and, by Lemma 12,  $(1-q)^{n-1} < ((n-1)q+1)^{-1}$ .

For  $r \in (0, \frac{2}{n+1})$ , the assumption that the maximum of (7) and thus the minimum in (6) are attained for  $q \in (0, 1)$  turns out to be correct, because the derivative of (7) is positive as  $q$  goes to 0 and negative as  $q$  goes to 1. Indeed

$$\begin{aligned}
&\lim_{q \rightarrow 0} \frac{r((nq+1)(1-q)^n - 1) + nq^2(1-q)^{n-1}}{q^2} \\
&= \lim_{q \rightarrow 0} \frac{-rnq(n+1)(1-q)^{n-1} + nq(2-(n+1)q)(1-q)^{n-2}}{2q} \\
&= \lim_{q \rightarrow 0} \frac{-r(n+1)n(1-q(n-1))(1-q)^{n-1}}{2} \\
&\quad + \frac{(2n-2n(n+1)q)(1-q)^{n-2} - (2nq-n(n+1)q^2)(n-2)(1-q)^{n-3}}{2} \\
&= \frac{2n-(n+1)nr}{2} > \frac{2n-(n+1)n\frac{2}{n+1}}{2} = 0
\end{aligned}$$

and

$$\lim_{q \rightarrow 1} \frac{r((nq+1)(1-q)^n - 1) + nq^2(1-q)^{n-1}}{q^2} = -r < 0.$$

We now claim that

$$T_n \geq \max_{r \in (0, \frac{2}{n+1})} \min_{q \in (0, 1)} \frac{((n-1)r+1)q}{(q+r(1-q))(1-(1-q)^n)}$$

$$\begin{aligned}
&= \max_{q \in (0,1)} \frac{\left( (n-1) \frac{nq^2(1-q)^{n-1}}{1-(nq+1)(1-q)^n} + 1 \right) q}{\left( q + \frac{nq^2(1-q)^n}{1-(nq+1)(1-q)^n} \right) (1 - (1-q)^n)} \\
&= \max_{q \in (0,1)} \frac{\frac{(n-1)nq^2(1-q)^{n-1} + 1 - (nq+1)(1-q)^n}{1-(nq+1)(1-q)^n} q}{\frac{q(1-(nq+1)(1-q)^n) + nq^2(1-q)^n}{1-(nq+1)(1-q)^n} (1 - (1-q)^n)} \\
&= \max_{q \in (0,1)} \frac{(n-1)nq^2(1-q)^{n-1} + 1 - (nq+1)(1-q)(1-q)^{n-1}}{(1 - (nq+1)(1-q)^n + nq(1-q)^n) (1 - (1-q)^n)} \\
&= \max_{q \in (0,1)} \frac{1 - (1-q + nq - n^2q^2)(1-q)^{n-1}}{(1 - (1-q)^n)^2} \\
&\geq \frac{1 - (1 - \frac{1}{n} + n\frac{1}{n} + n^2\frac{1}{n^2})(1 - \frac{1}{n})^n}{(1 - (1-q)^n)^2} = \frac{1}{1 - (1 - \frac{1}{n})^n}.
\end{aligned}$$

The first inequality holds because  $r$  is chosen from a smaller set and the minimum is attained for  $q \in (0, 1)$  when  $r \in (0, \frac{2}{n+1})$ . The first equality holds because (8) is satisfied at the minimum and a one-to-one correspondence exists between values  $q \in (0, 1)$  and  $r \in (0, \frac{2}{n+1})$ . The second inequality is then obtained by setting  $q = 1/n$ .

## H Proof of Theorem 11

To bound the maximum revenue from discriminatory prices, we can use the following result concerning the optimal prices for the welfare maximization problem.

**Lemma 17** (Gilbert and Mosteller [12], Section 5.a). *Let  $F = U[0, 1]$ . Then, for  $i \geq 10$ ,*

$$1 - \frac{2}{H_{i+1} + i + 1.5 - 0.310} \leq p_i^W \leq 1 - \frac{2}{H_{i+1} + i + 1.5 - 0.121},$$

where  $H_{i+1}$  is the  $(i+1)$ st harmonic number.

Since  $R_{n-1}^d = \varphi(p_{n-1}^R) = \varphi(p_n^W)$ , and since  $\varphi(x) = 2x - 1$  for the uniform distribution, the lemma implies that for  $n \geq 10$ ,

$$R_{n-1}^d \leq 1 - \frac{4}{H_{n+1} + n + 1.5 - 0.121}.$$

By an estimate of harmonic numbers due to Young [23],  $H_{n+1} \leq \log(n+1) + \gamma + \frac{1}{2(n+1)}$ , where  $\gamma \approx 0.57722$  is the Euler-Mascheroni constant. This implies that for  $n \geq 10$ ,

$$R_n^d \leq 1 - \frac{4}{\log(n+2) + \gamma + \frac{1}{2(n+2)} + n + 2.5 - 0.121} \leq \tilde{R}_n^d,$$

where  $\tilde{R}_n^d = 1 - \frac{4}{n + \log(n+2) + 3}$ . We further have that  $R_n^a = \max_{p \geq 0} p(1 - p^n)$ , which by first-order conditions implies  $p = \sqrt[n]{1/(n+1)}$  and thus  $R_n^a = \sqrt[n]{1/(n+1)}(1 - 1/(n+1))$ . Elementary calculus now shows that

$$\frac{d}{dn} \frac{\tilde{R}_n^d}{R_n^a} = (n+1)^{\frac{n+1}{n}} \frac{4n^2(n+3) - \log(n+1)(n+2)(n^2 + 2n - 3 + 2(n+1)\log(n+2) + \log^2(n+2))}{n^3(n+2)(n+3 + \log(n+2))^2},$$

which is non-positive if and only if

$$\log(n+1) \geq \frac{n+3}{n+2} \cdot \frac{4n^2}{n^2 + 2n - 3 + 2(n+1)\log(n+2) + \log^2(n+2)}.$$

Closer analysis reveals the right-hand side to be increasing with limit 4, so that  $\frac{d}{dn} tRd_n/R_n^a \leq 0$  for all  $n \geq e^3 \approx 20.09$  and thus

$$\max_{n \geq 20} \frac{R_n^d}{R_n^a} \leq \max_{n \geq 20} \frac{\tilde{R}_n^d}{R_n^a} = \frac{\tilde{R}_{20}^d}{R_{20}^a} \approx 1.0352.$$

By computing the values of  $R_n^d/R_n^a$  explicitly for  $n = 1, \dots, 19$ , we see that

$$\max_{n \leq 19} \frac{\tilde{R}_n^d}{R_n^a} = \frac{\tilde{R}_{11}^d}{R_{11}^a} \approx 1.0368,$$

as claimed.

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